Contents 35

Sets and Probability

Learning outcomes

In this Workbook you will learn about probability. In the first Section you will learn about sets and how they may be combined together using the operations of union and intersection. Then you will learn how to apply the notation of sets to the notion of probability and learn about the fundamental laws of probability.

Sets

35.1

Introduction

If we can identify a property which is common to several objects, it is often useful to group them together. Such a grouping is called a **set**. Engineers for example, may wish to study all components of a production run which fail to meet some specified tolerance. Mathematicians may look at sets of numbers with particular properties, for example, the set of all even numbers, or the set of all numbers greater than zero. In this block we introduce some terminology that is commonly used to describe sets, and practice using set notation. This notation will be particularly useful when we come to study probability in Section 35.2.

Prerequisites

Before starting this Section you should ...

- have knowledge of basic algebra

Learning Outcomes

On completion you should be able to ...

- state what is meant by a set
- use set notation
- explain the concepts of the intersection and union of two sets
- define what is meant by the complement of a set
- use Venn diagrams to illustrate sets

1. Sets

A **set** is any collection of objects. Here, the word 'object' is used in its most general sense: an object may be a diode, an aircraft, a number, or a letter, for example.
A set is often described by listing the collection of objects - these are the members or **elements** of the set. We usually write this list of elements in curly brackets, and denote the full set by a capital letter. For example,

$C = \{$the resistors produced in a factory on a particular day$\}$
$D = \{$on, off$\}$
$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

The elements of set C, above, are the resistors produced in a factory on a particular day. These could be individually labeled and listed individually but as the number is large it is not practical or sensible to do this. Set D lists the two possible states of a simple switch, and the elements of set E are the digits used in the decimal system.
Sometimes we can describe a set in words. For example,

'A is the set all odd numbers'.

Clearly all the elements of this set A cannot be listed.
Similarly,

'B is the set of binary digits' i.e. $B = \{0, 1\}$.

B has only two elements.

A set with a finite number of elements is called a **finite set**. B, C, D and E are finite sets. The set A has an infinite number of elements and so is not a finite set. It is called an **infinite set**.

Two sets are **equal** if they contain exactly the same elements. For example, the sets $\{9, 10, 14\}$ and $\{10, 14, 9\}$ are equal since the order in which elements are written is unimportant. Note also that repeated elements are ignored. The set $\{2, 3, 3, 3, 5, 5\}$ is equal to the set $\{2, 3, 5\}$.

Subsets

Sometimes one set is contained completely within another set. For example if $X = \{2, 3, 4, 5, 6\}$ and $Y = \{2, 3, 6\}$ then all the elements of Y are also elements of X. We say that Y is a **subset** of X and write $Y \subseteq X$.

Example 1

Given $A = \{0, 1, 2, 3\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$ and $C = \{0, 1\}$, state which sets are subsets of other sets.

Solution

A is a subset of B, that is $A \subseteq B$
C is a subset of B, that is $C \subseteq B$
C is a subset of A, that is $C \subseteq A$.

Task

A factory produces cars over a five day period; Monday to Friday. Consider the following sets,

(a) $A = \{\text{cars produced from Monday to Friday}\}$

(b) $B = \{\text{cars produced from Monday to Thursday}\}$

(c) $C = \{\text{cars produced on Friday}\}$

(d) $D = \{\text{cars produced on Wednesday}\}$

(e) $E = \{\text{cars produced on Wednesday or Thursday}\}$

State which sets are subsets of other sets.

Your solution

Answer

(a) B is a subset of A, that is, $B \subseteq A$.

(b) C is a subset of A, that is, $C \subseteq A$.

(c) D is a subset of A, that is, $D \subseteq A$.

(d) E is a subset of A, that is, $E \subseteq A$.

(e) D is a subset of B, that is, $D \subseteq B$.

(f) E is a subset of B, that is, $E \subseteq B$.

(g) D is a subset of E, that is, $D \subseteq E$.

The symbol $\in$

To show that an element belongs to a particular set we use the symbol $\in$. This symbol means **is a member of** or 'belongs to'. The symbol $\notin$ means **is not a member of** or 'does not belong to'.

For example if $X = \{$all even numbers$\}$ then we may write $4 \in X$, $6 \in X$, $7 \notin X$ and $11 \notin X$.

The empty set and the universal set

Sometimes a set will contain no elements. For example, suppose we define the set K by

$K = \{$all odd numbers which are divisible by 4$\}$

Since there are no odd numbers which are divisible by 4, then K has no elements. The set with no elements is called the **empty set**, and it is denoted by $\emptyset$.

On the other hand, the set containing all the objects of interest in a particular situation is called the **universal set**, denoted by S. The precise universal set will depend upon the context. If, for example, we are concerned only with whole numbers then $S = \{\cdots - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$. If we are concerned only with the decimal digits then $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The complement of a set

Given a set A and a universal set S we can define a new set, called the **complement** of A and denoted by A'. The complement of A contains all the elements of the universal set that are **not in** A.

Example 2

Given $A = \{2, 3, 7\}$, $B = \{0, 1, 2, 3, 4\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ state
(a) A' (b) B'

Solution

(a) The elements of A' are those which belong to S but not to A.

$A' = \{0, 1, 4, 5, 6, 8, 9\}$

(b) $B' = \{5, 6, 7, 8, 9\}$

Sometimes a set is described in a mathematical way. Suppose the set Q contains all numbers which are divisible by 4 and 7. We can write

$Q = \{x : x$ is divisible by 4 and x is divisible by 7$\}$

The symbol : stands for 'such that '. We read the above as 'Q is the set comprising all elements x, such that x is divisible by 4 and by 7'.

2. Venn diagrams

Sets are often represented pictorially by **Venn diagrams** (see Figure 1).

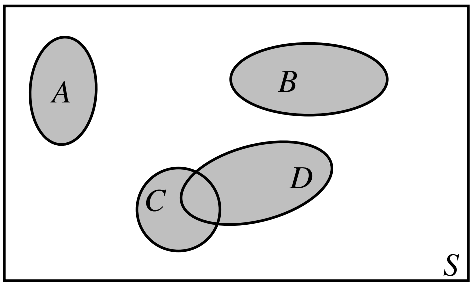


Figure 1

Here A, B, C, D represent sets. The sets A, B have no items in common so are drawn as non-intersecting regions whilst the sets C, D have some items in common so are drawn overlapping.
In a Venn diagram the universal set is represented by a rectangle and sets of interest by area regions within this rectangle.

Example 3

Represent the sets $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ using a Venn diagram.

Solution

The elements 0 and 1 are in set A, represented by the small circle in the diagram. The large circle represents set B and so contains the elements 0,1,2,3 and 4. A suitable universal set in this case is the set of all integers. The universal set is shown by the rectangle.

Note that $A \subseteq B$. This is shown in the Venn diagram by A being completely inside B.

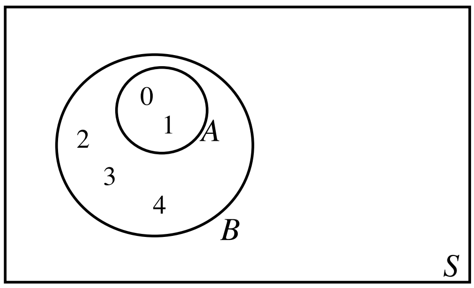


Figure 2: The set A is contained completely within B

Task

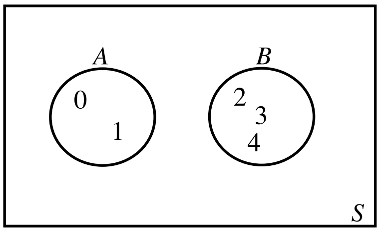
Given $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ draw Venn diagrams showing
(a) A and B (b) A' (c) B'

Your solution

(a)

Answer

Note that A and B have no elements in common. This is represented pictorially in the Venn diagram by circles which are totally separate from each other as shown in the diagram.

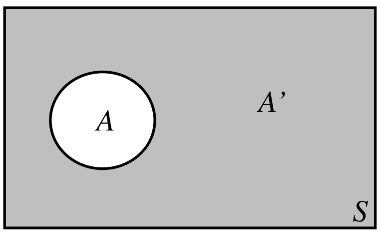


Your solution

(b)

Answer

The complement of A is the set whose elements do not belong to A. The set A' is shown shaded in the diagram.



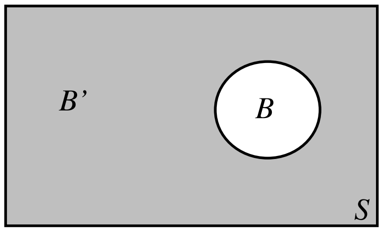
The complement of A contains elements which are not in A.

Your solution

(c)

Answer

The set B' is shown shaded in the diagram.



3. The intersection and union of sets

Intersection

Given two sets, A and B, the **intersection** of A and B is a set which contains elements that are common both to A and B. We write $A \cap B$ to denote the intersection of A and B. Mathematically we write this as:

Key Point 1

Intersection of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

This says that the intersection contains all the elements x such that x belongs to A and also x belongs to B.

Note that $A \cap B$ and $B \cap A$ are identical. The intersection of two sets can be represented by a Venn diagram as shown in Figure 3.

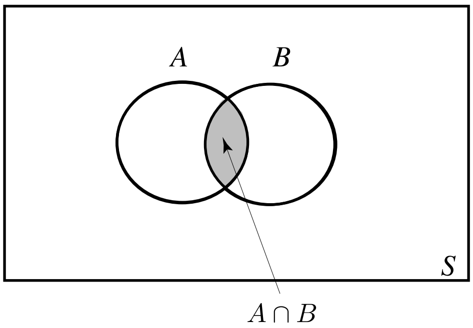


Figure 3: The overlapping area represents $A \cap B$

Example 4

Given $A = \{3, 4, 5, 6\}$, $B = \{3, 5, 9, 10, 15\}$ and $C = \{4, 6, 10\}$ state

(a) $A \cap B$, (b) $B \cap C$ and draw a Venn diagram representing these intersections.

Solution

(a) The elements common to both A and B are 3 and 5. Hence $A \cap B = \{3, 5\}$

(b) The only element common to B and C is 10. Hence $B \cap C = \{10\}$

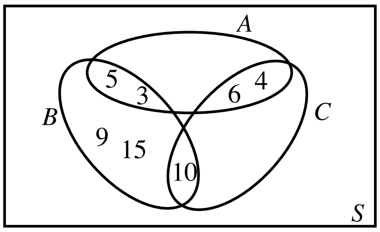


Figure 4

Task

Given $D = \{a, b, c\}$ and $F = \{\text{the entire alphabet}\}$ state $D \cap F$.

Your solution

Answer

The elements common to D and F are a, b and c, and so $D \cap F = \{a, b, c\}$

Note that D is a subset of F and so $D \cap F = D$.

The intersection of three or more sets is possible, and is the subject of the next Example.

Example 5

Given $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 3, 4, 7, 9\}$ state

(a) $A \cap B$ (b) $(A \cap B) \cap C$ (c) $B \cap C$ (d) $A \cap (B \cap C)$

Solution

(a) The elements common to A and B are 1, 2 and 3 so $A \cap B = \{1, 2, 3\}$.

(b) We need to consider the sets $(A \cap B)$ and C. $A \cap B$ is given in (a). The elements common to $(A \cap B)$ and C are 2 and 3. Hence $(A \cap B) \cap C = \{2, 3\}$.

(c) The elements common to B and C are 2, 3 and 4 so $B \cap C = \{2, 3, 4\}$.

(d) We look at the sets A and $(B \cap C)$. The common elements are 2 and 3. Hence $A \cap (B \cap C) = \{2, 3\}$.

Note from (b) and (d) that here $(A \cap B) \cap C = A \cap (B \cap C)$.

The example illustrates a general rule. For any sets A, B and C it is true that

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The position of the brackets is thus unimportant. They are usually omitted and we write $A \cap B \cap C$.

Suppose that sets A and B have no elements in common. Then their intersection contains no elements and we say that A and B are **disjoint** sets. We express this as

$$A \cap B = \emptyset$$

Recall that $\emptyset$ is the empty set. Disjoint sets are represented by separate area regions in the Venn diagram.

Union

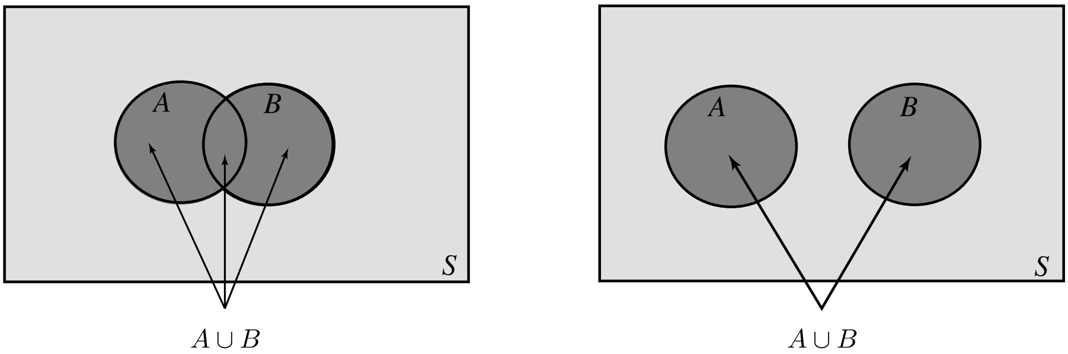
The **union** of two sets A and B is a set which contains all the elements of A together with all the elements of B. We write $A \cup B$ to denote the union of A and B. We can describe the set $A \cup B$ formally by:

Key Point 2

Union of Sets

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$$

Thus the elements of the set $A \cup B$ are those quantities x such that x is a member of A or a member of B or a member of both A and B. The deeply shaded areas of Figure 5 represents $A \cup B$.



(a) (b)

Figure 5

In Figure 5(a) the sets intersect, whereas in Figure 5(b) the sets have no region in common. We say they are **disjoint**.

Example 6

Given $A = \{0, 1\}$, $B = \{1, 2, 3\}$ and $C = \{2, 3, 4, 5\}$ write down

(a) $A \cup B$

(b) $A \cup C$

(c) $B \cup C$

Solution

(a) $A \cup B = \{0, 1, 2, 3\}$

(b) $A \cup C = \{0, 1, 2, 3, 4, 5\}$

(c) $B \cup C = \{1, 2, 3, 4, 5\}$.

Recall that there is no need to repeat elements in a set. Clearly the order of the union is unimportant so $A \cup B = B \cup A$.

Given $A = \{2,3,4,5,6\}$, $B = \{2,4,6,8,10\}$ and $C = \{3,5,7,9,11\}$ state
(a) $A \cup B$ (b) $(A \cup B) \cap C$ (c) $A \cap B$ (d) $(A \cap B) \cup C$ (e) $A \cup B \cup C$

Your solution

Answer

(a) $A \cup B = \{2,3,4,5,6,8,10\}$

(b) We need to look at the sets $(A \cup B)$ and C. The elements common to both of these sets are 3 and 5. Hence $(A \cup B) \cap C = \{3,5\}$.

(c) $A \cap B = \{2,4,6\}$

(d) We consider the sets $(A \cap B)$ and C. We form the union of these two sets to obtain $(A \cap B) \cup C = \{2,3,4,5,6,7,9,11\}$.

(e) The set formed by the union of all three sets will contain all the elements from all the sets:

$$A \cup B \cup C = \{2,3,4,5,6,7,8,9,10,11\}$$

Exercises

1. Given a set A, its complement A' and a universal set S, state which of the following expressions are true and which are false.

   (a) $A \cup A' = S$ (b) $A \cap S = \emptyset$ (c) $A \cap A' = \emptyset$
   (d) $A \cap A' = S$ (e) $A \cup \emptyset = S$ (f) $A \cup \emptyset = A$
   (g) $A \cup \emptyset = \emptyset$ (h) $A \cap \emptyset = A$ (i) $A \cap \emptyset = \emptyset$
   (j) $A \cup S = A$ (k) $A \cup S = \emptyset$ (l) $A \cup S = S$

2. Given $A = \{a, b, c, d, e, f\}$, $B = \{a, c, d, f, h\}$ and $C = \{e, f, x, y\}$ obtain the sets:

   (a) $A \cup B$ (b) $B \cap C$ (c) $A \cap (B \cup C)$
   (d) $C \cap (B \cup A)$ (e) $A \cap B \cap C$ (f) $B \cup (A \cap C)$

3. List the elements of the following sets:

   (a) $A = \{x : x$ is odd and x is greater than 0 and less than 12$\}$
   (b) $B = \{x : x$ is even and x is greater than 19 and less than 31$\}$

4. Given $A = \{5, 6, 7, 9\}$, $B = \{0, 2, 4, 6, 8\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ list the elements of each of the following sets:

   (a) A' (b) B' (c) $A' \cup B'$
   (d) $A' \cap B'$ (e) $A \cup B$ (f) $(A \cup B)'$
   (g) $(A \cap B)'$ (h) $(A' \cap B)'$ (i) $(B' \cup A)'$

   What do you notice about your answers to (c),(g)?

   What do you notice about your answers to (d),(f)?

5. Given that A and B are intersecting sets, i.e. are not disjoint, show on a Venn diagran the following sets

   (a) A' (b) B' (c) $A \cup B'$ (d) $A' \cup B'$ (e) $A' \cap B'$

Answers

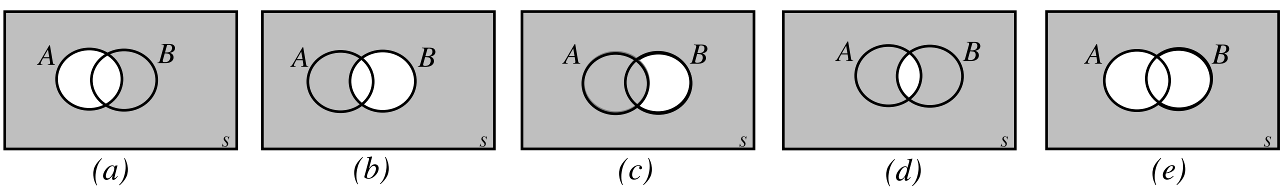
1.(a) T, (b) F, (c) T, (d) F, (e) F, (f) T, (g) F, (h) F, (i) T, (j) F, (k) F), (l) T.

2.(a) $\{a, b, c, d, e, f, h\}$, (b) $\{f\}$, (c) $\{a, c, d, e, f\}$, (d) $\{e, f\}$, (e) $\{f\}$, (f) $\{a, c, d, e, f, h\}$.

3.(a) $\{1, 3, 5, 7, 9, 11\}$, (b) $\{20, 22, 24, 26, 28, 30\}$.

4.(a) $\{0, 1, 2, 3, 4, 8\}$, (b) $\{1, 3, 5, 7, 9\}$, (c) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (d) $\{1, 3\}$,
(e) $\{0, 2, 4, 5, 6, 7, 8, 9\}$, (f) $\{1, 3\}$, (g) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (h) $\{1, 3, 5, 6, 7, 9\}$,
(i) $\{0, 2, 4, 8\}$.

5.



Elementary Probability

35.2

Introduction

Probability is about the study of uncertainty. Engineers are expected to design and produce systems which are both useful and reliable. Essentially we are dealing with situations where 'chance' is at work and probability theory gives us the theoretical underpinning necessary for a full understanding of any experimental results we observe in practice. Probability theory also gives us the tools to set up mathematical models of systems and processes which are affected by random occurrences or 'chance'. In fact the study of probability enables engineers to discuss the reliability of the processes they use and the systems they produce in terms that other engineers, scientists and designers can understand. It is worth noting that 'chance' is taken to be responsible for variations in simple manufactured products such as screws, bolts, and light bulbs as well as complex products such as cars, ships and aircraft. In each of these products, small chance variations in raw materials and production processes may have a substantial effect on a product.

Prerequisites

Before starting this Section you should ...

- understand the ideas of sets and subsets

Learning Outcomes

On completion you should be able to ...

- explain the terms 'random experiment' and 'event'
- calculate the probability of an event occurring
- calculate the probability that an event does not occur

1. Introductory probability

Probability as an informal idea is something you will have been familiar with for a long time. In conversation with friends, you must have used sentences such as

- 'It might start raining soon'
- 'I might be lucky and pass all my examinations'
- 'It is very unlikely that my team will not win the Premiership this year'
- 'Getting a good degree will improve my chances of getting a good job'

Essentially, when you are talking about whether some event is likely to happen, you are using the concept of probability. In reality, we need to agree on some terminology so that misunderstanding may be avoided.

Terminology

To start with there are four terms – experiment, outcome, event and sample space – that need formal definition. There will, of course, be others as you progress through this Workbook.

1. **Experiment:** - an activity with an observable result, or set of results, for example

   (a) tossing a coin, the result being a Head or a Tail

   (b) testing a component, the result being a defective or non-defective component

   (c) maximum speed testing of standard production cars;

   (d) testing to destruction armour plating intended for use on tanks.

   Some of the experiments outlined above have a very limited set of results (tossing a coin) while others (destruction testing) may give a widely variable set of results. Also it is worth noting that destruction testing is not appropriate for all products. Companies manufacturing say trucks or explosives could not possibly test to destruction on a large scale - they would have little or nothing left to market!

2. **Outcome** - an outcome is simply an observable result of an experiment, for example

   (a) tossing a coin, the possible outcomes are Heads or Tails

   (b) testing a component, the outcome being a defective or non-defective component

   (c) maximum speed testing of standard production cars, the outcomes being a set of numbers representing the maximum speeds of a set of vehicles

   (d) testing to destruction armour plating intended for use on tanks, the outcomes might be (for example) the numbers of direct hits sustained before destruction.

3. **Event** - this is just an outcome or set of outcomes to an experiment of interest to the experimenter.

4. **Sample Space** - a sample space is the set of all possible outcomes of an experiment.

For example, if we throw a die then the sample space is $\{1, 2, 3, 4, 5, 6\}$ and two possible events are

(a) a score of 3 or more, represented by the set: $\{3, 4, 5, 6\}$

(b) a score which is even, represented by the set: $\{2, 4, 6\}$.

Everyday examples include games of chance.

Example 7

Obtain the sample space of the experiment *throwing a single coin.*

Solution

Consider the experiment of throwing a coin which can land Heads up (H) or Tails up (T). We list the outcomes as a set $\{H,\ T\}$ − the order being unimportant. $\{H, T\}$ is the sample space. On any particular throw of a coin, Heads or Tails are equally likely to occur. We say that, for a **fair** coin, H and T are **equally likely outcomes**.

If the sample space can be written in the form of a list (possibly infinite) then it is called a **discrete sample space** (e.g. number of tosses of a fair coin before Heads occurs). If this is not possible then it is called a **continuous sample space** (e.g. positions where shells land in a tank battle).

Task

List the equally likely outcomes to the experiments:
(a) throwing a fair die with six faces labelled 1 to 6
[Note: 'die' is the singular of 'dice', although most people use 'dice' instead.]

(b) throwing three fair coins.

Your solution

Answer

(a) $\{1, 2, 3, 4, 5, 6\}$ (b) $\{TTT,\ TTH,\ THT,\ THH,\ HTT,\ HTH,\ HHT,\ HHH\}$

Task

For the following list of experiments, list (if possible) a suitable sample space. If you cannot write out a suitable sample space, describe one in words.

(a) Test a light switch

(b) Count the daily traffic accidents in Loughborough involving cyclists

(c) Measure the tensile strength of small gauge steel wire

(d) Test the maximum current carrying capacity of household mains cabling

(e) Test the number of on-off switchings that a new type of fluorescent tube will cope with before failure

(f) Pressure test an underwater TV camera.

Your solution

Answer

Sample spaces might be

(a) {works, fails}

(b) {0,1, 2, 3, . . . }, hopefully a small upper limit!

(c) {Suitable continuous range (0 $\rightarrow$) depending on the wire}

(d) {Suitable continuous range (0 $\rightarrow$) depending on the type of cable}

(e) {0,1, 2, 3, ... }, hopefully a high upper limit!

(f) {Suitable continuous range (0 $\rightarrow$) }

Example 8

A car manufacturer offers certain options on its family cars. Customers may order:

(a) either automatic gearboxes or manual gearboxes

(b) either sunroof or air-conditioning

(c) either steel wheels or allow wheels

(d) either solid colour paint or metallic paint

Find the number of outcomes in the sample space of options that it is possible to order and represent them using a suitable diagram.

Solution

A suitable diagram is shown in Figure 6. The diagram makes it easy to find the number of outcomes simply by counting. It also points the way to a formula for calculating the number of outcomes.

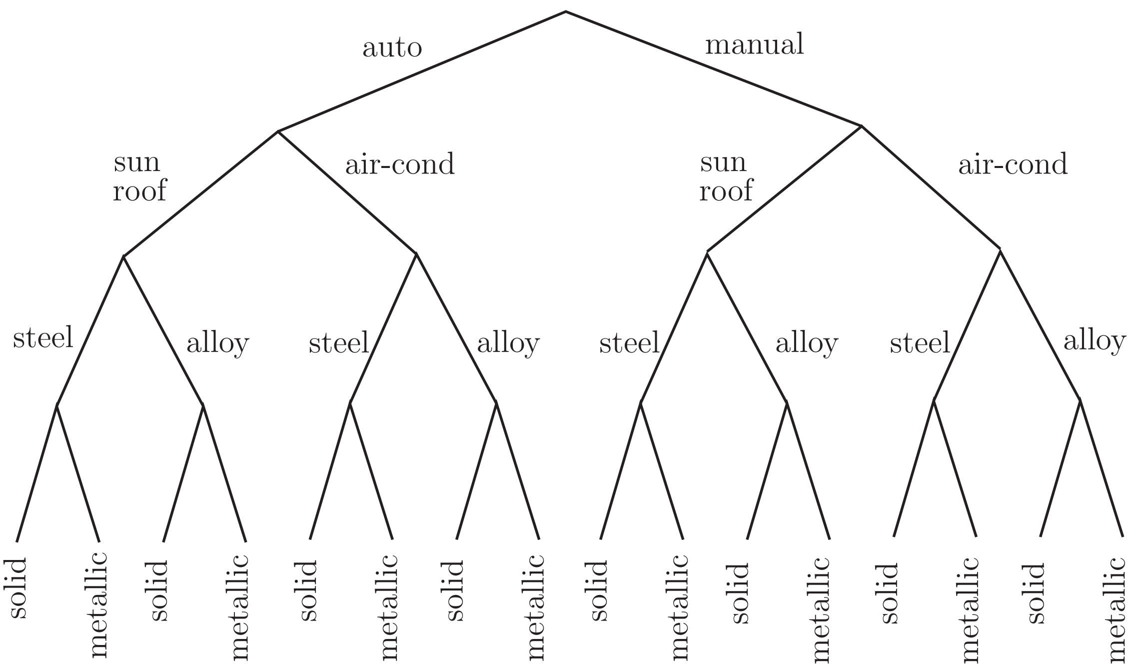


Figure 6: Tree diagram

In this case there is a total of 16 outcomes in the sample space of options. Note that in each case the customer makes two choices. This implies that there are

$$2 \times 2 \times 2 \times 2 = 16$$

options in total.

Diagrams such as the one above are called **tree diagrams**. They are only suitable in simple situations.

Events

As we have already noted a collection of some or all of the outcomes of an experiment is called an **event**. So an event is a **subset** of the sample space. For example, if we throw a die then the sample space is $\{1, 2, 3, 4, 5, 6\}$ and two possible events are

(a) a score of 3 or more, represented by the set: $\{3, 4, 5, 6\}$

(b) a score which is even, represented by the set: $\{2, 4, 6\}$.

Example 9

Two coins are thrown. List the ordered outcomes for the event when just one Tail is obtained.

Solution

$\{H, T\}, \ \{T, H\}$

Note that here the **order does matter** unlike for sets in general.

Task

Three coins are thrown. List the ordered outcomes which belong to each of the following events.

(a) two Tails are obtained
(b) at least two Tails are obtained
(c) at most two Tails are obtained

State the relationship between (a) and (b) and that between (a) and (c).

Your solution

Answer

(a) $\{TTH, \ THT, \ HTT\}$
(b) $\{TTT, \ TTH, \ THT, \ HTT\}$
(c) $\{TTH, \ THT, \ HTT, \ THH, \ HTH, \ HHT, \ HHH\}$

(a) is a subset of (b) and (a) is also a subset of (c).

Task

A new type of paint to be used in the manufacture of garden equipment is tested for impact shock resistance to damage and scratch resistance to damage. The results (50 samples) are as follows

		Shock Resistance	
		Good	**Poor**
Scratch Resistance	**Good**	20	15
	Poor	12	3

If A is the event {High Shock Resistance} and B is the event {High Scratch Resistance}, describe the following events and determine the number of samples in each event.

(a) $A \cup B$ (b) $A \cap B$ (c) A' (d) B'

Your solution

Answer

(a) The event $A \cup B$ consists of those samples which have either good shock and good scratch resistance (or both).

$$n(A \cup B) = 47.$$

(b) The event $A \cap B$ consists of those samples which have both good shock and good scratch resistance.

$$n(A \cap B) = 20.$$

(c) The event A' consists of those samples which do not have good shock resistance.

$$n(A') = 18.$$

(d) The event B' consists of those samples which do not have good scratch resistance.

$$n(B') = 15.$$

Complement

We have met the complement before (Section 35.1 page 5) in relation to sets. We consider it again here in relation to sample spaces and events. The **complement** of an event is the set of outcomes which are not members of the event.
For example, the experiment of throwing a 6-faced die has sample space $S = \{1, 2, 3, 4, 5, 6\}$.

The event "score of 3 or more is obtained" is the set $\{3, 4, 5, 6\}$.

The complement of this event is $\{1, 2\}$ which can be described in words as "score of 3 or more is **not** obtained" or "score of 1 or 2 is obtained".

The event: "even score is obtained" is the set $\{2, 4, 6\}$.

The complement of this event is $\{1, 3, 5\}$ or, in words " even score is **not** obtained" or "odd score is obtained".

In the last but one Task concerning tossing three coins:

- the complement of event (a) is $\{TTT,\ THH,\ HTH,\ HHT,\ HHH\}$,
- the complement of event (b) is $\{THH,\ HTH,\ HHT,\ HHH\}$
- the complement of event (c) is $\{TTT\}$.

State, in words, what are the complements of each of the following events in relation to the experiment of throwing three coins (avoid using the word *not*):
(a) two Heads are obtained (b) at least two Heads are obtained (c) at most two Heads are obtained.

Your solution

Answer
(a) no Heads, one Head or three Heads

(b) no Heads or one Head

(c) three Heads

Notation: It is customary to use a capital letter to denote an event. For example, $A = \{$two Heads are thrown$\}$. The complementary event is denoted A'.

Hence, in the case where $A = \{$at least two Heads are thrown$\}$, A' is the event $\{$fewer than two Heads are thrown$\}$.

2. Definitions of probability

Relative frequency applied to probability

Consider the experiment of throwing a single coin many times.

Suppose we throw a coin 10 times and obtain six Heads and four Tails; does this suggest that the coin is biased? Clearly not! What about the case when we obtain 9 Heads and 1 Tail?
We conducted an experiment in which a coin was thrown 100 times and the result recorded each time as 1 if a Head appeared face up and 0 if a Tail appeared. In Figure 7 we have plotted the average score $\dfrac{r}{n}$, where r is the number of Heads and n is the total number of throws, against n for $n = 10, 20, \ldots, 100$. The quantity $\dfrac{r}{n}$ is called the **relative frequency** of Heads.

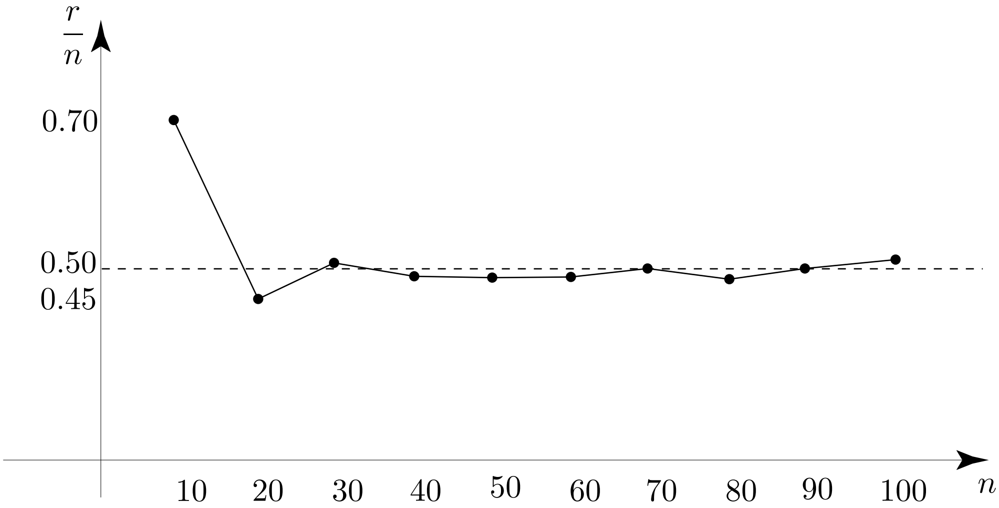


Figure 7

As n increases the relative frequency settles down near the value $\dfrac{1}{2}$. This is an experimental estimate of the probability of throwing a Head with this particular coin. Note that when $n = 50$ this estimate was 0.49 and when $n = 100$ this estimate was 0.51. When we repeated the whole experiment again, the value of $\dfrac{r}{n}$ when $n = 100$ was 0.46. Hence the use of the word **estimate**. Normally, as the number of trials is increased the estimate tends to settle down but this is not certain to occur.
Theoretically, the probability of obtaining a Head when a fair coin is thrown is $\dfrac{1}{2}$. Experimentally, we **expect** the relative frequency to approach $\dfrac{1}{2}$ as n increases.

Equi-probable spaces and the principle of equally likely outcomes

An equi-probable space is a sample space in which the chance that any one sample point occurs is equal to the chance that any other sample point occurs. Whether a sample space is an equi-probable space is usually determined by inspection or logic.

(a) Tossing a coin: the sample space S is, using an obvious notation:

$S = \{H, T\}$

H and T are the two simple outcomes and are equally likely to occur.

(b) Rolling a die: a sample space is:

$S = \{1, 2, 3, 4, 5, 6\}$

Each number from 1 to 6 is a simple outcome and is equally likely to occur.

(c) Tossing three coins: a sample space comprised of simple outcomes is:

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Each of the eight outcomes stated is equally likely to occur.

(d) Counting the number of heads when three coins are tossed. Here $S = \{$3 Heads, 2 Heads, 1 Head, 0 Heads$\}$. This is **not** an equi-probable space and the outcomes are **not** equally likely. (For example: the event $\{$2 Heads$\}$ is the union of three simple events $\{HHT, HTH, THH\}$ so must occur more often than the event $\{HHH\}$.

Of the sample spaces above all are equi-probable spaces except for (d).

Key Point 3

The Principle of Equally Likely Outcomes

This states that each simple outcome in an equi-probable space is equally likely to occur. This principle enables us to deduce the probabilities that simple events (and hence more complicated events which are combinations of simple events) occur.

Notation

If A is an event associated with a sample space S the the probability of A occurring is denoted by $\mathsf{P}(A)$.

Referring to the examples above we may immediately deduce that

(a) $\mathsf{P}\{H\} = \mathsf{P}\{T\} = \frac{1}{2}$

(b) $\mathsf{P}\{1\} = \mathsf{P}\{2\} = \mathsf{P}\{3\} = \mathsf{P}\{4\} = \mathsf{P}\{5\} = \mathsf{P}\{6\} = \frac{1}{6}$

(c) $\mathsf{P}\{HHH\} = \mathsf{P}\{HHT\} = \mathsf{P}\{HTH\} = \mathsf{P}\{THH\} =$

$$\mathsf{P}\{HTT\} = \mathsf{P}\{THT\} = \mathsf{P}\{TTH\} = \mathsf{P}\{TTT\} = \frac{1}{8}$$

Definition

We can now define probability using the Principle of Equally Likely Outcomes as follows:
If a sample space S consists of n simple outcomes which are equally likely and an event A consists of m of those simple outcomes, then

$$\mathsf{P}(A) = \frac{m}{n} = \frac{\text{the number of simple outcomes in } A}{\text{the number of simple outcomes in S}}$$

It follows from this definition that $0 \leq \mathsf{P}(A) \leq 1$.

- If $\mathsf{P}(A) = 1$ we say that the event A is **certain** because A is identical to S.
- If $\mathsf{P}(A) = 0$ we say that the event A is **impossible** because A is empty.

The set with no outcomes in it is called the **empty set** and written $\emptyset$; therefore $\mathsf{P}(\emptyset) = 0$.

Task

For each of the following events $A,\ B,\ C$, list and count the number of outcomes it contains and hence calculate the probability of $A,\ B$ or C occurring.

(a) $A =$ "throwing 3 or higher with one die",

(b) $B =$ "throwing exactly two Heads with three coins",

(c) $C =$ "throwing a total score of 14 with two dice".

Your solution

Answer

(a) There are six possible equally likely outcomes of the experiment and four of them, $\{3, 4, 5, 6\}$, constitute the event A; hence $\mathsf{P}(A) = \frac{4}{6} = \frac{2}{3}$.

(b) There are eight equally likely outcomes of which three, $\{HHT,\ HTH,\ THH\}$ are elements of B; hence $\mathsf{P}(B) = \frac{3}{8}$.

(c) It is impossible to throw a total higher than 12 so that $C = \emptyset$ and $\mathsf{P}(C) = 0$.

Not surprisingly, the probabilities of an event A and its complement are related. The probability of the event A' is easily found from the identity

$$\frac{\text{number of outcomes in } A}{\text{total number of outcomes}} + \frac{\text{number of outcomes not in } A}{\text{total number of outcomes}} \equiv 1$$

so that $\mathsf{P}(A) + \mathsf{P}(A') \equiv 1$

Key Point 4

The Complement Rule

$$\mathsf{P}(A') = 1 - \mathsf{P}(A)$$

In words:

The probability of the complement of A occurring
is equal to 1 minus the probability of A occurring.

Task

For the events in (a) and (b) of the previous Task find $\mathsf{P}(A')$ and $\mathsf{P}(B')$ and describe in words what A' and B' are in this case.

Your solution

Answer

(a) $\mathsf{P}(A) = \frac{2}{3}$ so that $\mathsf{P}(A') = \frac{1}{3}$.

A' is the event of throwing a score of less than 3 on one die.

(b) $\mathsf{P}(B) = \frac{3}{8}$ so that $\mathsf{P}(B') = \frac{5}{8}$

B' is the event of throwing no Heads, exactly one Head or exactly three Heads with three coins.

The use of the event A' can sometimes simplify the calculation of the probability $\mathsf{P}(A)$. For example, suppose that two dice are thrown and we require the probability of the event

A: that we obtain a total score of at least four.

There are many combinations that produce a total score of at least four; however there are only 3 combinations that produce a total score of two or three which is the complementary event to the one of interest. The event $A' = \{(1,1),\ (1,2),\ (2,1)\}$ (where we use an obvious notation of stating the total score on the first die followed by the score on the second die) is the complement of A.

Now $\mathsf{P}(A') = \frac{3}{36}$ since there are 6×6 possible combinations in throwing two dice. Thus

$$\mathsf{P}(A) = 1 - \mathsf{P}(A') = 1 - \frac{3}{36} = \frac{33}{36} = \frac{11}{12}.$$

Task

Find the probability of obtaining a total score of at least five when three dice are thrown. Hint: identify A and A', then calculate $\mathsf{P}(A')$, then $\mathsf{P}(A)$.

Your solution

Answer

There are $6 \times 6 \times 6 = 216$ possible outcomes. If, for example, $(1, 3, 6)$ denotes the scores of 1 on die one, 3 on die two and 6 on die three and if A is the event 'a total score of five or more' then A' is the event 'a total score of less than 5' i.e.

$A' = \{(1, 1, 1),\ (2, 1, 1),\ (1, 2, 1),\ (1, 1, 2)\}$

There are four outcomes in A' and hence $\mathsf{P}(A') = \dfrac{4}{216}$ so that $\mathsf{P}(A) = \dfrac{212}{216}$.

Exercises

1. For each of the following experiments, state whether the variable is discrete or continuous. In each case state the sample space.

   (a) The number of defective items in a batch of twenty is noted.

   (b) The weight, in kg, of lubricating oil drained from a machine is determined using a spring balance.

   (c) The natural logarithm of the weight, in kg, according to a spring balance, of lubricating oil drained from a machine, is noted.

2. An experiment consists of throwing two four-faced dice (regular tetrahedra) with faces labelled 1, 2, 3, 4.

   (a) Write down the sample space of this experiment.

   (b) If A is the event 'total score is at least 4' list the outcomes belonging to A'.

   (c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?

   (d) What is the probability that a double: i.e. $(1, 1), (2, 2), (3, 3), (4, 4)$ will **not** be thrown?

   (e) What is the probability that a double is not thrown and the score is less than 6?

3. A **lot** consists of 10 good articles, 4 articles with minor defects and 2 with major defects. One article is chosen at random from the lot. Find the probability that:

   (a) it has no defects,

   (b) it has no major defects,

   (c) it is either good or has major defects.

4. Propeller shafts for marine applications are inspected to ensure that they satisfy both diameter requirements and surface finish requirements. The results of 400 inspections are as follows:

		Diameter Requirements	
		Good	**Poor**
Surface Finish	**Good**	200	50
	Poor	80	70

   (a) What is the probability that a shaft selected at random satisfies the surface finish requirements?

   (b) What is the probability that a shaft selected at random satisfies both diameter and surface finish requirements?

   (c) What is the probability that a shaft selected at random satisfies either the diameter or the surface finish requirements?

   (d) What is the probability that a shaft selected at random satisfies neither the diameter nor the surface finish requirements?

Answers

1. (a) The variable is discrete. The sample space is $\{1, 2, \ldots, 20\}$.

   (b) The variable is continuous. The sample space is the set of real numbers x such that $0 \leq x < \infty$.

   (c) The variable is continuous. The sample space is the set of real numbers x such that $-\infty < x < \infty$.

2. (a) $S = \{(1,1),(1,2),(1,3),(1,4)$

   $$(2,1),(2,2),(2,3),(2,4)$$

   $$(3,1),(3,2),(3,3),(3,4)$$

   $$(4,1),(4,2),(4,3),(4,4)\}$$

   (b) $A' = \{(1,1),(1,2),(2,1)\}$

   (c) The outcomes in the event are $\{(2,4),(3,3),(3,4),(4,2),(4,3),(4,4)\}$ so the probability of this event occurring is $\dfrac{6}{16} = \dfrac{3}{8}$. The probability of the complement event is $1-\dfrac{3}{8} = \dfrac{5}{8}$.

   (d) The probability of a double occurring is $\dfrac{4}{16}$ so the probability of the complement (i.e, double not thrown) is $1 - \dfrac{4}{16} = \dfrac{3}{4}$.

   (e) Here, consider the sample space in (a). If the doubles and those outcomes with a score greater than 6 are removed we have left the event :

   $$\{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}.$$

   Hence the probability of this event occurring is $\dfrac{8}{16} = \dfrac{1}{2}$.

3. Let G be the event 'article is good' , M_n be the event 'article has minor defect' and M_j be the event 'article has major defect'

   (a) Here we require P(G). Obviously P$(G) = \dfrac{10}{16} = \dfrac{5}{8}$

   (b) We require P$(M_j') = 1-$ P$(M_j) = 1 - \dfrac{2}{16} = \dfrac{7}{8}$

   (c) The event we require is the complement of the event M_n.

   Since P$(M_n) = \dfrac{4}{16} = \dfrac{1}{4}$ we have P$(M_n') =$ P$(G$ or $M_j) = 1 - \dfrac{1}{4} = \dfrac{3}{4}$.

   Equivalently P(G) + P$(M_n) = \dfrac{10}{16} + \dfrac{2}{16} = \dfrac{12}{16} = \dfrac{3}{4}$

4. (a) $\dfrac{250}{400} = 0.625$ (b) $\dfrac{200}{400} = 0.5$ (c) $\dfrac{330}{400} = 0.825$ (d) $\dfrac{70}{400} = 0.175$

Addition and Multiplication Laws of Probability

35.3

Introduction

When we require the probability of two events occurring simultaneously or the probability of one or the other or both of two events occurring then we need probability laws to carry out the calculations. For example, if a traffic management engineer looking at accident rates wishes to know the probability that cyclists and motorcyclists are injured during a particular period in a city, he or she must take into account the fact that a cyclist and a motorcyclist might collide. (Both events can happen simultaneously.)

Prerequisites

Before starting this Section you should ...

- understand the ideas of sets and subsets
- understand the concepts of probability and events

Learning Outcomes

On completion you should be able to ...

- state and use the addition law of probability
- define the term independent events
- state and use the multiplication law of probability
- understand and explain the concept of conditional probability

1. The addition law

As we have already noted, the sample space S is the set of all possible outcomes of a given experiment. Certain events A and B are subsets of S. In the previous Section we defined what was meant by $\mathsf{P}(A)$, $\mathsf{P}(B)$ and their complements in the particular case in which the experiment had equally likely outcomes.
Events, like sets, can be combined to produce new events.

- $A \cup B$ denotes the event that event A or event B (or both) occur when the experiment is performed.
- $A \cap B$ denotes the event that both A and B occur together.

In this Section we obtain expressions for determining the probabilities of these combined events, which are written $\mathsf{P}(A \cup B)$ and $\mathsf{P}(A \cap B)$ respectively.

Types of events

There are two types of events you will need to able to identify and work with: mutually exclusive events and independent events. (We deal with independent events in subsection 3.)

Mutually exclusive events

Mutually exclusive events are events that by definition cannot happen together. For example, when tossing a coin, the events 'head' and 'tail' are mutually exclusive; when testing a switch 'operate' and 'fail' are mutually exclusive; and when testing the tensile strength of a piece of wire, 'hold' and 'snap' are mutually exclusive. In such cases, the probability of both events occurring together must be zero. Hence, using the usual set theory notation for events A and B, we may write:

$$\mathsf{P}(A \cap B) = 0, \qquad \text{provided that } A \text{ and } B \text{ are mutually exclusive events}$$

Task

Decide which of the following pairs of events (A and B) arising from the experiments described are mutually exclusive.

(a) Two cards are drawn from a pack

$A = \{$a red card is drawn$\}$
$B = \{$a picture card is drawn$\}$

(b) The daily traffic accidents in Loughborough involving pedal cyclists and motor cyclists are counted

$A = \{$three motor cyclists are injured in collisions with cars$\}$
$B = \{$one pedal cyclist is injured when hit by a bus$\}$

(c) A box contains 20 nuts. Some have a metric thread, some have a British Standard Fine (BSF) thread and some have a British Standard Whitworth (BSW) thread.

$A = \{$first nut picked out of the box is BSF$\}$
$B = \{$second nut picked out of the box is metric $\}$

Your solution

Answer

(a) A and B are not mutually exclusive.

(b) A and B are mutually exclusive.

(c) A and B are not mutually exclusive.

Key Point 5

The Addition Law of Probability - Simple Case

If two events A and B are mutually exclusive then

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$$

Key Point 6

The Addition Law of Probability - General Case

If two events are A and B then

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$$

If $A \cap B = \emptyset$, i.e. A and B are mutually exclusive, then $\mathsf{P}(A \cap B) = \mathsf{P}(\emptyset) = 0$, and this general expression reduces to the simpler case.
This rule can be extended to three or more events, for example:

$$\mathsf{P}(A \cup B \cup C) = \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) - \mathsf{P}(A \cap B) - \mathsf{P}(A \cap C) - \mathsf{P}(B \cap C) + \mathsf{P}(A \cap B \cap C)$$

Example 10

Consider a pack of 52 playing cards. A card is selected at random. What is the probability that the card is **either** a diamond **or** a ten?

Solution

If A is the event $\{$*a diamond is selected*$\}$ and B is the event $\{$*a ten is selected*$\}$ then obviously $\mathsf{P}(A) = \dfrac{13}{52}$ and $\mathsf{P}(B) = \dfrac{4}{52}$. The intersection event $A \cap B$ consists of only one member - the ten of diamonds - which gets counted twice hence $\mathsf{P}(A \cap B) = \dfrac{1}{52}$.

Therefore $\mathsf{P}(A \cup B) = \dfrac{13}{52} + \dfrac{4}{52} - \dfrac{1}{52} = \dfrac{16}{52}$.

Task

A bag contains 20 balls, 3 are coloured red, 6 are coloured green, 4 are coloured blue, 2 are coloured white and 5 are coloured yellow. One ball is selected at random. Find the probabilities of the following events.

(a) the ball is either red or green

(b) the ball is not blue

(c) the ball is either red or white or blue. (Hint: consider the complementary event.)

Your solution

Answer

Note that a ball has only one colour, designated by the letters R, G, B, W, Y.

(a) $\mathsf{P}(R \cup G) = \mathsf{P}(R) + \mathsf{P}(G) = \frac{3}{20} + \frac{6}{20} = \frac{9}{20}$.

(b) $\mathsf{P}(B') = 1 - \mathsf{P}(B) = 1 - \frac{4}{20} = \frac{16}{20} = \frac{4}{5}$.

(c) The complementary event is $G \cup Y$, $\mathsf{P}(G \cup Y) = \frac{6}{20} + \frac{5}{20} = \frac{11}{20}$.

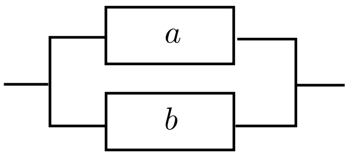
Hence $\mathsf{P}(R \cup W \cup B) = 1 - \frac{11}{20} = \frac{9}{20}$

In the last Task part (c) we could alternatively have used an obvious extension of the law of addition for mutually exclusive events:

$$\mathsf{P}(R \cup W \cup B) = \mathsf{P}(R) + \mathsf{P}(W) + \mathsf{P}(B) = \frac{3}{20} + \frac{2}{20} + \frac{4}{20} = \frac{9}{20}.$$

Task

The diagram shows a simplified circuit in which two independent components a and b are connected in parallel.



The circuit functions if either or both of the components are operational. It is known that if A is the event 'component a is operating' and B is the event 'component b is operating' then $\mathsf{P}(A) = 0.99$, $\mathsf{P}(B) = 0.98$ and $\mathsf{P}(A \cap B) = 0.9702$. Find the probability that the circuit is functioning.

Your solution

Answer

The probability that the circuit is functioning is $\mathsf{P}(A \cup B)$. In words: either a or b or both must be functioning if the circuit is to function. Using the keypoint:

$$\begin{aligned} \mathsf{P}(A \cup B) &= \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B) \\ &= 0.99 + 0.98 - 0.9702 = 0.9998 \end{aligned}$$

Not surprisingly the probability that the circuit functions is greater than the probability that either of the individual components functions.

Exercises

1. The following people are in a room: 5 men aged 21 and over, 4 men under 21, 6 women aged 21 and over, and 3 women under 21. One person is chosen at random. The following events are defined: $A = \{\text{the person is aged 21 and over}\}$; $B = \{\text{the person is under 21}\}$; $C = \{\text{the person is male}\}$; $D = \{\text{ the person is female}\}$. Evalute the following:

   (a) $\mathsf{P}(B \cup D)$

   (b) $\mathsf{P}(A' \cap C')$
   Express the meaning of these events in words.

2. A card is drawn at random from a deck of 52 playing cards. What is the probability that it is an ace or a face card (i.e. K, Q, J)?

3. In a single throw of two dice, what is the probability that neither a double nor a sum of 9 will appear?

Answers

1. (a) $\mathsf{P}(B \cup D) = \mathsf{P}(B) + \mathsf{P}(D) - \mathsf{P}(B \cap D)$

$$\mathsf{P}(B) = \frac{7}{18}, \quad \mathsf{P}(D) = \frac{9}{18}, \quad \mathsf{P}(B \cap D) = \frac{3}{18}$$

$$\therefore \quad \mathsf{P}(B \cup D) = \frac{7}{18} + \frac{9}{18} - \frac{3}{18} = \frac{13}{18}$$

(b) $\mathsf{P}(A' \cap C')$ $\qquad A' = \{\text{people under 21}\}$ $\quad C' = \{\text{people who are female}\}$

$$\therefore \qquad \mathsf{P}(A' \cap C') = \frac{3}{18} = \frac{1}{6}$$

2. $F = \{\text{face card}\}$ $\quad A = \{\text{card is ace}\}$ $\quad \mathsf{P}(F) = \frac{12}{52}, \quad \mathsf{P}(A) = \frac{4}{52}$

$$\therefore \qquad \mathsf{P}(F \cup A) = \mathsf{P}(F) + \mathsf{P}(A) - \mathsf{P}(F \cap A) = \frac{12}{52} + \frac{4}{52} - 0 = \frac{16}{52}$$

3. $D = \{\text{double is thrown}\}$ $\qquad N = \{\text{sum is } 9\}$

$\mathsf{P}(D) = \frac{6}{36}$ (36 possible outcomes in an experiment in which all the outcomes are equally probable).

$$\mathsf{P}(N) = \mathsf{P}\{(6 \cap 3) \cup (5 \cap 4) \cup (4 \cap 5) \cup (3 \cap 6)\} = \frac{4}{36}$$

$$\mathsf{P}(D \cup N) = \mathsf{P}(D) + \mathsf{P}(N) - \mathsf{P}(D \cap N) = \frac{6}{36} + \frac{4}{36} - 0 = \frac{10}{36}$$

$$\mathsf{P}((D \cup N)') = 1 - \mathsf{P}(D \cup N) = 1 - \frac{10}{36} = \frac{26}{36}$$

2. Conditional probability - dependent events

Suppose a bag contains 6 balls, 3 red and 3 white. Two balls are chosen (without replacement) at random, one after the other. Consider the two events R, W:

R is event "first ball chosen is red"
W is event "second ball chosen is white"

We easily find $\text{P}(R) = \frac{3}{6} = \frac{1}{2}$. However, determining the probability of W is not quite so straightforward. If the first ball chosen is red then the bag subsequently contains 2 red balls and 3 white. In this case $\text{P}(W) = \frac{3}{5}$. However, if the first ball chosen is white then the bag subsequently contains 3 red balls and 2 white. In this case $\text{P}(W) = \frac{2}{5}$. What this example shows is that the probability that W occurs is clearly **dependent** upon whether or not the event R has occurred. The probability of W occurring is **conditional** on the occurrence or otherwise of R.

The **conditional probability** of an event B occurring given that event A has occurred is written $\text{P}(B|A)$. In this particular example

$$\text{P}(W|R) = \frac{3}{5} \qquad \text{and} \qquad \text{P}(W|R') = \frac{2}{5}.$$

Consider, more generally, the performance of an experiment in which the outcome is a member of an event A. We can therefore say that the event A has occurred. What is the probability that B then occurs? That is what is $\text{P}(B|A)$? In a sense we have a new sample space which is the event A. For B to occur some of its members must also be members of event A. So, for example, in an equi-probable space, $\text{P}(B|A)$ must be the number of outcomes in $A \cap B$ divided by the number of outcomes in A. That is

$$\text{P}(B|A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}.$$

Now if we divide both the top and bottom of this fraction by the total number of outcomes of the experiment we obtain an expression for the conditional probability of B occurring given that A has occurred:

Key Point 7

Conditional Probability

$$\text{P}(B|A) = \frac{\text{P}(A \cap B)}{\text{P}(A)} \qquad \text{or, equivalently} \qquad \text{P}(A \cap B) = \text{P}(B|A)\text{P}(A)$$

To illustrate the use of conditional probability concepts we return to the example of the bag containing 3 red and 3 white balls in which we consider two events:

- R is event "first ball is red"
- W is event "second ball is white"

Let the red balls be numbered 1 to 3 and the white balls 4 to 6. If, for example, $(3, 5)$ represents the fact that the first ball is 3 (red) and the second ball is 5 (white) then we see that there are $6 \times 5 = 30$ possible outcomes to the experiment (no ball can be selected twice).

If the first ball is red then only the fifteen outcomes $(1, x)$, $(2, y)$, $(3, z)$ are then possible (here $x \neq 1$, $y \neq 2$ and $z \neq 3$). Of these fifteen, the six outcomes $\{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (3,1),\ (3,2)\}$ will produce the required result, i.e. the event in which both balls chosen are red, giving a probability: $\mathsf{P}(B|A) = \dfrac{6}{15} = \dfrac{2}{5}$.

Example 11

A box contains six 10 Ω resistors and ten 30 Ω resistors. The resistors are all unmarked and are of the same physical size.

(a) One resistor is picked at random from the box; find the probability that:

(i) It is a 10 Ω resistor.

(ii) It is a 30 Ω resistor.

(b) At the start, two resistors are selected from the box. Find the probability that:

(i) Both are 10 Ω resistors.

(ii) The first is a 10 Ω resistor and the second is a 30 Ω resistor.

(iii) Both are 30 Ω resistors.

Solution

(a) (i) As there are six 10 Ω resistors in the box that contains a total of 6 + 10 = 16 resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\mathsf{P}(10\ \Omega) = \frac{6}{16} = \frac{3}{8}$$

(ii) As there are ten 30 Ω resistors in the box that contains a total of 6 + 10 = 16 resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\mathsf{P}(30\ \Omega) = \frac{10}{16} = \frac{5}{8}$$

(b) (i) As there are six 10 Ω resistors in the box that contains a total of 6 + 10 = 16 resistors, and there is an *equally likely chance* of any resistor being selected, then

$$\mathsf{P}(\text{first selected is a 10 } \Omega \text{ resistor}) = \frac{6}{16} = \frac{3}{8}$$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are only five 10 Ω resistors left in the box which now contains 5 + 10 = 15 resistors.

Hence, $\mathsf{P}(\text{second selected is also a 10 } \Omega \text{ resistor}) = \dfrac{5}{15} = \dfrac{1}{3}$

And, $\mathsf{P}(\text{both are 10 } \Omega \text{ resistors}) = \dfrac{3}{8} \times \dfrac{1}{3} = \dfrac{1}{8}$

Solution (contd.)

(b) (ii) As before, P(first selected is a 10 Ω resistor) $= \dfrac{6}{18} = \dfrac{3}{8}$

If the first resistor selected was a 10 Ω one, then when the second resistor is selected, there are still ten 30 Ω resistors left in the box which now contains 5 + 10 = 15 resistors. Hence,

P(second selected is a 30 Ω resistor) $= \dfrac{10}{15} = \dfrac{2}{3}$

And, P(first was a 10 Ω resistor and second was a 30 Ω resistor) $= \dfrac{3}{8} \times \dfrac{2}{3} = \dfrac{1}{4}$

(b) (iii) As there are ten 30 Ω resistors in the box that contains a total of 6 + 10 = 16 resistors, and there is an *equally likely chance* of any resistor being selected, then

P(first selected is a 30 Ω resistor) $= \dfrac{10}{16} \times \dfrac{5}{8}$

If the first resistor selected was a 30 Ω one, then when the second resistor is selected, there are only nine 30 Ω resistors left in the box which now contains 5 + 10 = 15 resistors.

Hence, P(second selected is also a 30 Ω resistor) $= \dfrac{9}{15} = \dfrac{3}{5}$

And, P(both are 30 Ω resistors) $= \dfrac{5}{8} \times \dfrac{3}{5} = \dfrac{3}{8}$

3. Independent events

If the occurrence of one event A does not affect, nor is affected by, the occurrence of another event B then we say that A and B are **independent** events. Clearly, if A and B are independent then

$$\mathsf{P}(B|A) = \mathsf{P}(B) \qquad \text{and} \qquad \mathsf{P}(A|B) = \mathsf{P}(A)$$

Then, using the Key Point 7 formula $\mathsf{P}(A \cup B) = \mathsf{P}(B|A)\mathsf{P}(A)$ we have, for independent events:

Key Point 8

The Multiplication Law

If A and B are independent events then

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) \times \mathsf{P}(B)$$

In words

'The probability of independent events A and B occurring is the product of the probabilities of the events occurring separately.'

In Figure 8 two components a and b are connected in series.

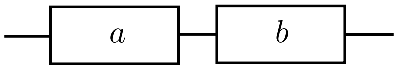


Figure 8

Define two events

- A is the event 'component a is operating'
- B is the event 'component b is operating'

Previous testing has indicated that $\mathsf{P}(A) = 0.99$, and $\mathsf{P}(B) = 0.98$. The circuit functions only if a and b are both operating simultaneously. The components are assumed to be independent.
Then the probability that the circuit is operating is given by

$$\mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B) = 0.99 \times 0.98 = 0.9702$$

Note that this probability is smaller then either $\mathsf{P}(A)$ or $\mathsf{P}(B)$.

Task

Decide which of the following pairs (A and B) of events arising from the experiments described are independent.

(a) One card is drawn from each of two packs

$A = \{$a red card is drawn from pack 1$\}$
$B = \{$a picture card is drawn from pack 2$\}$

(b) The daily traffic accidents in Hull involving pedal cyclists and motor cyclists are counted

$A = \{$three motor cyclists are injured in separate collisions with cars$\}$
$B = \{$one pedal cyclist is injured when hit by a bus$\}$

(c) Two boxes contains 20 nuts each, some have a metric thread, some have a British Standard Fine (BSF) threads and some have a British Standard Whitworth (BSW) thread. A nut is picked out of each box.

$A = \{$nut picked out of the first box is BSF$\}$
$B = \{$nut picked out of the second box is metric $\}$

(d) A box contains 20 nuts, some have a metric thread, some have a British Standard Fine (BSF) threads and some have British Standard Whitworth (BSW) thread. Two nuts are picked out of the box.

$A = \{$first nut picked out of the box is BSF$\}$
$B = \{$second nut picked out of the box is metric $\}$

Your solution

Answer

(a), (b), (c): A and B are independent. (d) A and B are not independent.

Key Point 9

Laws of Elementary Probability

Let a sample space S consist of the n simple distinct events $E_1, E_2 \ldots E_n$ and let A and B be events contained in S.

Then:

(a) $0 \leq \mathsf{P}(A) \leq 1$. $\mathsf{P}(A) = 0$ is interpreted as meaning that the event A cannot occur and $\mathsf{P}(A) = 1$ is interpreted as meaning that the event A is certain to occur.

(b) $\mathsf{P}(A) + \mathsf{P}(A') = 1$ where the event A' is the complement of the event A

(c) $\mathsf{P}(E_1) + \mathsf{P}(E_2) + \cdots + \mathsf{P}(E_n) = 1$ where $E_1, E_2, \ldots E_n$ form the sample space

(d) If A and B are *any* two events then $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$

(e) If A and B are two *mutually exclusive* events then $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$

(f) If A and B are two *independent* events then $\mathsf{P}(A \cap B) = \mathsf{P}(A) \times \mathsf{P}(B)$.

Example 12

A circuit has three independent switches A, B and C wired in parallel as shown in the figure below.

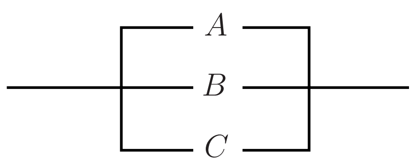


Figure 9

Current can only flow through the bank of switches if at least one of them is closed. The probability that any given switch is closed is 0.9. Calculate the probability that current can flow through the bank of switches.

Solution

Assume that A is the event {switch A is closed}. Similarly for switches B and C. We require $\mathsf{P}(A \cup B \cup C)$, the probability that at least one switch is closed. Using set theory,

$$\begin{aligned}\mathsf{P}(A \cup B \cup C) &= \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) - [\mathsf{P}(A \cap B) + \mathsf{P}(B \cap C) + \mathsf{P}(C \cap A)] \\ &\quad + \mathsf{P}(A \cap B \cap C)\end{aligned}$$

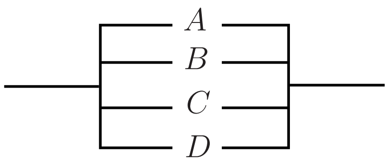
Using the fact that the switches operate independently,

$$\begin{aligned}\mathsf{P}(A \cup B \cup C) &= 0.9 + 0.9 + 0.9 - [0.9 \times 0.9 + 0.9 \times 0.9 + 0.9 \times 0.9] \\ &\quad + 0.9 \times 0.9 \times 0.9 \\ &= 2.7 - 2.43 + 0.729 = 0.999\end{aligned}$$

Note that the result implies that the system is more likely to allow current to flow than any single switch in the system. This is why replication is built into systems requiring a high degree of reliability such as aircraft control systems.

Task

A circuit has four independent switches A, B, C and D wired in parallel as shown in the diagram below.



Current can only flow through the bank of switches if at least one of them is closed. The probabilities that switches A, B, C and D are closed are 0.9, 0.8, 0.7 and 0.6 respectively. Calculate the probability that current can flow through the bank of switches.

Answer

Denoting the switches by A, B, C and D we have:

$$\begin{aligned}&\mathsf{P}(A \cup B \cup C \cup D) \\ &\quad = \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) + \mathsf{P}(D) \\ &\qquad - \mathsf{P}(A \cap B) - \mathsf{P}(B \cap C) - \mathsf{P}(C \cap D) - \mathsf{P}(D \cap A) - \mathsf{P}(A \cap C) - \mathsf{P}(B \cap D) \\ &\qquad + \mathsf{P}(A \cap B \cap C) + \mathsf{P}(B \cap C \cap D) + \mathsf{P}(C \cap D \cap A) + \mathsf{P}(D \cap A \cap B) \\ &\qquad - \mathsf{P}(A \cap B \cap C \cap D)\end{aligned}$$

Using the fact that the switches operate independently and substituting gives:

$$\mathsf{P}(A \cup B \cup C \cup D) = 3 - 3.35 + 1.65 - 0.3024 = 0.9976$$

Hence, the probability that current can flow through the bank of switches is 0.9976.

Exercises

1. A box contains 4 bad tubes and 6 good tubes. Two are drawn out together. One of them is tested and found to be good. What is the probability that the other one is also good?

2. A man owns a house in town and a cottage in the country. In any one year the probability of the town house being burgled is 0.01 and the probability of the country cottage being burgled is 0.05. In any one year what is the probability that:

   (a) both will be burgled? (b) one or the other (but not both) will be burgled ?

3. In a Baseball Series, teams A and B play until one team has won 4 games. If team A has probability 2/3 of winning against B in a single game, what is the probability that the Series will end only after 7 games are played?

4. The probability that a single aircraft engine will fail during flight is q. A multi-engine plane makes a successful flight if at least half its engines run. Assuming that the engines operate independently, find the values of q for which a two-engine plane is to be preferred to a four-engine plane.

5. Current flows through a relay only if it is closed. The probability of any relay being closed is 0.95. Calculate the probability that a current will flow through a circuit composed of 3 relays in parallel. What assumption must be made?

6. A central heating installation and maintenance engineer keeps a record of the causes of failure of systems he is called out to repair. The causes of failure are classified as 'electrical', 'gas', or in some cases 'other'. A summary of the records kept of failures involving either gas or electrical faults is as follows:

		Electrical	
		Yes	**No**
Gas	**Yes**	53	11
	No	23	13

   (a) Find the probability that failure involves gas given that it involves electricity.

   (b) Find the probability that failure involves electricity given that it involves gas.

Answers

1. Let $G_i = \{i^{th}$ tube is good$\}$ $\quad B_i = \{i^{th}$ tube is bad$\}$

   $\mathsf{P}(G_2|G_1) = \frac{5}{9}$ (only 5 good tubes left out of 9).

2. (a) $H = \{$house is burgled$\}$ $\;C = \{$cottage is burgled$\}$

   (b) $\mathsf{P}(H \cap C) = \mathsf{P}(H)\mathsf{P}(C) = (0.01)(0.05) = 0.0005$ since events independent

   $$\begin{aligned}\mathsf{P}(\text{one or the other (but not both)}) &= \mathsf{P}((H \cap C') \cup (H' \cap C)) = \mathsf{P}(H \cap C') + \mathsf{P}(H' \cap C)\\ &= \mathsf{P}(H)\mathsf{P}(C') + \mathsf{P}(H')\mathsf{P}(C)\\ &= (0.01)(0.95) + (0.99)(0.05) = 0.059.\end{aligned}$$

3. Let A_i be event $\{A$ wins the i^{th} game$\}$

   required event is $\underbrace{\{A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') \cap (\ldots)}_{\text{no. of ways of arranging 3 in 6 i.e. } {}^6C_3}$

   $\mathsf{P}(\text{required event}) = {}^6C_3\mathsf{P}(A_1 \cap A_2 \cap A_3 \cap A_4' \cap A_5' \cap A_6') = {}^6C_3[\mathsf{P}(A_1)]^3\mathsf{P}[A_1']^3 = \frac{160}{729}$

4. Let E_i be event $\{i^{th}$ engine success$\}$

   Two-engine plane: flight success if $\{(E_1 \cap E_2) \cup (E'_1 \cap E_2)(E_1 \cap E'_2)\}$ occurs

   $$\begin{aligned}\mathsf{P}(\text{required event}) &= \mathsf{P}(E_1)\mathsf{P}(E_2) + \mathsf{P}(E_1')\mathsf{P}(E_2) + \mathsf{P}(E_1)\mathsf{P}(E_2')\\ &= (1-q)^2 + 2q(1-q) = 1 - q^2\end{aligned}$$

   Four-engine plane: success if following event occurs

   $$\underbrace{\{E_1 \cap E_2 \cap E_3' \cap E_4'\}}_{{}^4C_2 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4'\}}_{{}^4C_1 \text{ ways}} \cap \underbrace{\{E_1 \cap E_2 \cap E_3 \cap E_4\}}_{{}^4C_0 \text{ ways}}$$

   required probability= $6(1-q)^2q^2 + 4(1-q)^3q + (1-q)^4 = 3q^4 - 4q^3 + 1$

   Two-engine plane is preferred if

   $$1 - q^2 > 3q^4 - 4q^3 + 1 \qquad \text{i.e. if} \quad 0 > q^2(3q-1)(q-1)$$

   Let $y = (3q-1)(q-1)$. By drawing a graph of this quadratic you will quickly see that a two-engine plane is preferred if $\frac{1}{3} < q < 1$.

Answers

5. Let A be event {relay A is closed}: Similarly for $B,\ C$

required event is $\{A \cap B \cap C\} \cup \underbrace{\{A' \cap B \cap C\}}_{^3C_1} \cap \underbrace{\{A' \cap B' \cap C\}}_{^3C_2}$

P(required event) $= (0.95)^3 + 3(0.95)^2(0.05) + 3(0.95)(0.05)^2 = 0.999875$

(or $1-$ P(all relays open) $= 1 - (0.05)^3 = 0.999875$.)

The assumption is that relays operate independently.

6 (a) A total of 76 failures involved electrical faults. Of the 76 some 53 involved gas. Hence

$$\text{P}\{\text{Gas Failure} \mid \text{Electrical Failure}\} = \frac{53}{76} = 0.697$$

6 (b) A total of 64 failures involved electrical faults. Of the 64 some 53 involved gas. Hence

$$\text{P}\{\text{Electrical Failure} \mid \text{Gas Failure}\} = \frac{53}{64} = 0.828$$

Total Probability and Bayes' Theorem

35.4

Introduction

When the ideas of probability are applied to engineering (and many other areas) there are occasions when we need to calculate conditional probabilities other than those already known. For example, if production runs of ball bearings involve say, four machines, we might know the probability that any given machine produces faulty ball bearings. If we are inspecting the total output prior to distribution to users, we might need to know the probability that a faulty ball bearing came from a particular machine. Even though we do not address the area of statistics known as Bayesian Statistics here, it is worth noting that Bayes' theorem is the basis of this branch of the subject.

Prerequisites

Before starting this Section you should ...

- understand the ideas of sets and subsets.
- understand the concepts of probability and events.
- understand the addition and multiplication laws and the concept of conditional probability.

Learning Outcomes

On completion you should be able to ...

- understand the term 'partition of a sample space'
- understand the special case of Bayes' theorem arising when a sample space is partitioned by a set and its complement
- be able to apply Bayes' theorem to solve basic engineering related problems

1. The theorem of total probability

To establish this result we start with the definition of a partition of a sample space.

A partition of a sample space

The collection of events $A_1, A_2, \ldots A_n$ is said to **partition** a sample space S if

(a) $A_1 \cup A_2 \cup \cdots \cup A_n = S$

(b) $A_i \cap A_j = \emptyset \quad$ for all i, j

(c) $A_i \neq \emptyset \quad$ for all i

In essence, a partition is a collection of non-empty, non-overlapping subsets of a sample space whose union is the sample space itself. The definition is illustrated by Figure 10.

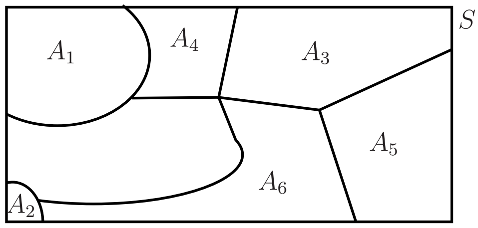


Figure 10

If B is any event within S then we can express B as the union of subsets:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n)$$

The definition is illustrated in Figure 11 in which an event B in S is represented by the shaded region.

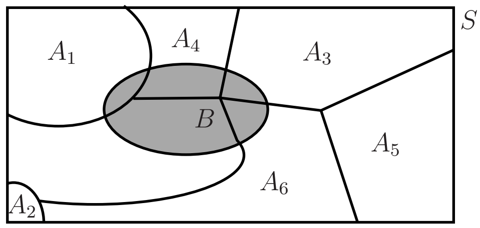


Figure 11

The bracketed events $(B \cap A_1), (B \cap A_2) \ldots (B \cap A_n)$ are mutually exclusive (if one occurs then none of the others can occur) and so, using the addition law of probability for mutually exclusive events:

$$\mathsf{P}(B) = \mathsf{P}(B \cap A_1) + \mathsf{P}(B \cap A_2) + \cdots + \mathsf{P}(B \cap A_n)$$

Each of the probabilities on the right-hand side may be expressed in terms of conditional probabilities:

$$\mathsf{P}(B \cap A_i) = \mathsf{P}(B|A_i)\mathsf{P}(A_i) \qquad \text{for all } i$$

Using these in the expression for $\mathsf{P}(B)$, above, gives:

$$\begin{aligned} \mathsf{P}(B) &= \mathsf{P}(B|A_1)\mathsf{P}(A_1) + \mathsf{P}(B|A_2)\mathsf{P}(A_2) + \cdots + \mathsf{P}(B|A_n)\mathsf{P}(A_n) \\ &= \sum_{i=1}^{n} \mathsf{P}(B|A_i)\mathsf{P}(A_i) \end{aligned}$$

This is the theorem of Total Probability. A related theorem with many applications in statistics can be deduced from this, known as Bayes' theorem.

2. Bayes' theorem

We again consider the conditional probability statement:

$$\mathsf{P}(A|B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B|A_1)\mathsf{P}(A_1) + \mathsf{P}(B|A_2)\mathsf{P}(A_2) + \cdots + \mathsf{P}(B|A_n)\mathsf{P}(A_n)}$$

in which we have used the theorem of Total Probability to replace $\mathsf{P}(B)$. Now

$$\mathsf{P}(A \cap B) = \mathsf{P}(B \cap A) = \mathsf{P}(B|A) \times \mathsf{P}(A)$$

Substituting this in the expression for $\mathsf{P}(A|B)$ we immediately obtain the result

$$\mathsf{P}(A|B) = \frac{\mathsf{P}(B|A) \times \mathsf{P}(A)}{\mathsf{P}(B|A_1)\mathsf{P}(A_1) + \mathsf{P}(B|A_2)\mathsf{P}(A_2) + \cdots + \mathsf{P}(B|A_n)\mathsf{P}(A_n)}$$

This is true for *any* event A and so, replacing A by A_i gives the result, known as Bayes' theorem as

$$\mathsf{P}(A_i|B) = \frac{\mathsf{P}(B|A_i) \times \mathsf{P}(A_i)}{\mathsf{P}(B|A_1)\mathsf{P}(A_1) + \mathsf{P}(B|A_2)\mathsf{P}(A_2) + \cdots + \mathsf{P}(B|A_n)\mathsf{P}(A_n)}$$

3. Special cases

In the case where we consider A to be an event in a sample space S (the sample space is partitioned by A and A') we can state simplified versions of the theorem of Total Probability and Bayes theorem as shown below.

The theorem of total probability: special case

This special case enables us to find the probability that an event B occurs taking into account the fact that another event A may or may not have occurred.
The theorem becomes

$$\mathsf{P}(B) = \mathsf{P}(B|A) \times \mathsf{P}(A) + \mathsf{P}(B|A') \times \mathsf{P}(A')$$

The result is easily seen by considering the general result already derived or it may be derived directly as follows. Consider Figure 12:

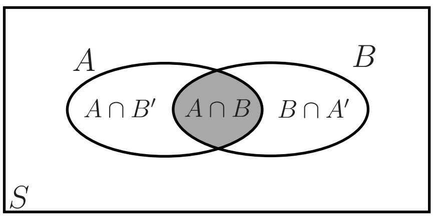


Figure 12

It is easy to see that the event B consists of the union of the (disjoint) events $A \cap B$ and $B \cap A'$ so that we may write B as the union of these disjoint events. We have

$$B = (A \cap B) \cup (B \cap A')$$

Since the events $A \cap B$ and $B \cap A'$ are disjoint, they must be independent and so

$$\mathsf{P}(B) = \mathsf{P}(A \cap B) + \mathsf{P}(B \cap A')$$

Using the conditional probability results we already have we may write

$$\begin{aligned}\mathsf{P}(B) &= \mathsf{P}(A\cap B)+\mathsf{P}(B\cap A') \\ &= \mathsf{P}(B\cap A)+\mathsf{P}(B\cap A') \\ &= \mathsf{P}(B|A)\times\mathsf{P}(A)+\mathsf{P}(B|A')\times\mathsf{P}(A')\end{aligned}$$

The result we have derived is

$$\mathsf{P}(B)=\mathsf{P}(B|A)\times\mathsf{P}(A)+\mathsf{P}(B|A')\times\mathsf{P}(A')$$

Bayes' theorem: special case

This result is obtained by supposing that the sample space S is partitioned by event A and its complement A' to give:

$$\mathsf{P}(A|B)=\frac{\mathsf{P}(B|A)\times\mathsf{P}(A)}{\mathsf{P}(B|A)\times\mathsf{P}(A)+\mathsf{P}(B|A')\times\mathsf{P}(A')}$$

Example 13

At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman?

Solution

Let M={Student is Male}, F={Student is Female}.
Note that M and F partition the sample space of students.
Let T={Student is over 6 feet tall}.

We know that $\mathsf{P}(M)=2/5$, $\mathsf{P}(F)=3/5$, $\mathsf{P}(T|M)=4/100$ and $\mathsf{P}(T|F)=1/100$.

We require $\mathsf{P}(F|T)$. Using Bayes' theorem we have:

$$\begin{aligned}\mathsf{P}(F|T) &= \frac{\mathsf{P}(T|F)\mathsf{P}(F)}{\mathsf{P}(T|F)\mathsf{P}(F)+\mathsf{P}(T|M)\mathsf{P}(M)} \\ &= \frac{\dfrac{1}{100}\times\dfrac{3}{5}}{\dfrac{1}{100}\times\dfrac{3}{5}+\dfrac{4}{100}\times\dfrac{2}{5}} \\ &= \frac{3}{11}\end{aligned}$$

Example 14

A factory production line is manufacturing bolts using three machines, A, B and C. Of the total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. It is known from previous experience with the machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from

(a) machine A (b) machine B (c) machine C?

Solution

Let

D={bolt is defective},

A={bolt is from machine A},

B={bolt is from machine B},

C={bolt is from machine C}.

We know that $\mathsf{P}(A) = 0.25$, $\mathsf{P}(B) = 0.35$ and $\mathsf{P}(C) = 0.4$.

Also

$\mathsf{P}(D|A) = 0.05, \mathsf{P}(D|B) = 0.04, \mathsf{P}(D|C) = 0.02.$

A statement of Bayes' theorem for three events A, B and C is

$$\begin{aligned}
\mathsf{P}(A|D) &= \frac{\mathsf{P}(D|A)\mathsf{P}(A)}{\mathsf{P}(D|A)\mathsf{P}(A) + \mathsf{P}(D|B)\mathsf{P}(B) + \mathsf{P}(D|C)\mathsf{P}(C)} \\
&= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.362
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathsf{P}(B|D) &= \frac{0.04 \times 0.35}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.406 \\
\mathsf{P}(C|D) &= \frac{0.02 \times 0.4}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.232
\end{aligned}$$

Task

An engineering company advertises a job in three newspapers, A, B and C. It is known that these papers attract undergraduate engineering readerships in the proportions 2:3:1. The probabilities that an engineering undergraduate sees and replies to the job advertisement in these papers are 0.002, 0.001 and 0.005 respectively. Assume that the undergraduate sees only one job advertisement.

(a) If the engineering company receives only one reply to it advertisements, calculate the probability that the applicant has seen the job advertised in place A.

(i) A, (ii) B, (iii) C.

(b) If the company receives two replies, what is the probability that both applicants saw the job advertised in paper A?

Your solution

Answer
Let

$A = \{\text{Person is a reader of paper } A\}$,

$B = \{\text{Person is a reader of paper } B\}$,

$C = \{\text{Person is a reader of paper } C\}$,

$R = \{\text{Reader applies for the job}\}$.

We have the probabilities

(a)

$$\begin{aligned} \mathsf{P}(A) &= 1/3 \qquad \mathsf{P}(R|A) = 0.002 \\ \mathsf{P}(B) &= 1/2 \qquad \mathsf{P}(R|B) = 0.001 \\ \mathsf{P}(C) &= 1/6 \qquad \mathsf{P}(R|C) = 0.005 \\ \mathsf{P}(A|R) &= \frac{\mathsf{P}(R|A)\mathsf{P}(A)}{\mathsf{P}(R|A)\mathsf{P}(A) + \mathsf{P}(R|B)\mathsf{P}(B) + \mathsf{P}(R|C)\mathsf{P}(C)} = \frac{1}{3} \end{aligned}$$

Similarly

$$\mathsf{P}(B|R) = \frac{1}{4} \qquad \text{and} \qquad \mathsf{P}(C|R) = \frac{5}{12}$$

(b) Now, assuming that the replies and readerships are independent

$$\begin{aligned} \mathsf{P}(\text{Both applicants read paper} A) &= \mathsf{P}(A|R) \times \mathsf{P}(A|R) \\ &= \frac{1}{3} \times \frac{1}{3} \\ &= \frac{1}{9} \end{aligned}$$

Exercises

1. Obtain the sample space of an experiment that consists of a fair coin being tossed four times. Consider the following events:

   A is the event 'all four results are the same.'

   B is the event 'exactly one Head occurs.'

   C is the event 'at least two Heads occur.'

   Show that $\mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) = \dfrac{17}{16}$ and explain why $\mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) > 1$.

2. The table below show the number of complete years a group of people have been working in their current employment.

Years of Employment	Number of People
0 or 1 year	15
2 or 3 years	12
4 or 5 years	9
6 or 8 years	6
8 to 11 years	6
12 years and over	2

   What is the probability that a person from the group, selected at random;

   (a) is in the modal group
   (b) has been working there for less than 4 years
   (c) has been working there for at least 8 years.

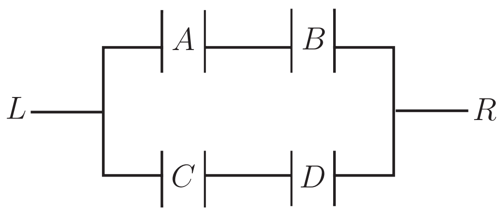
3. It is a fact that if A and B are independent events then it is also true that A' and B' are independent events. If A and B are independent events such that the probability that they both occur simultaneously is $\dfrac{1}{8}$ and the probability that neither of them will occur is $\dfrac{3}{8}$, find:

   (a) the probability that event A will occur
   (b) the probability that event B will occur.

4. If A and B are two events associated with an experiment and $\mathsf{P}(A) = 0.4$, $\mathsf{P}(A \cup B) = 0.7$ and $\mathsf{P}(B) = p$, find:

   (a) the choice of p for which A and B are mutually exclusive
   (b) the choice of p for which A and B are independent.

5. The probability that each relay closes in the circuit shown below is p. Assuming that each relay functions independently of the others, find the probability that current can flow from L to R.



6. From a batch of 100 items of which 20 are defective, exactly two items are chosen, one at a time, without replacement. Calculate the probabilities that:

   (a) the first item chosen is defective

   (b) both items chosen are defective

   (c) the second item chosen is defective.

7. A garage mechanic keeps a box of good springs to use as replacements on customers cars. The box contains 5 springs. A colleague, thinking that the springs are for scrap, tosses three faulty springs into the box. The mechanic picks two springs out of the box while servicing a car. Find the probability that:

   (a) the first spring drawn is faulty (b) the second spring drawn is faulty.

8. Two coins are tossed. Find the conditional probability that two Heads will occur given that at least one occurs.

9. Machines A and B produce 10% and 90% respectively of the production of a component intended for the motor industry. From experience, it is known that the probability that machine A produces a defective component is 0.01 while the probability that machine B produces a defective component is 0.05. If a component is selected at random from a day's production and is found to be defective, find the probability that it was made by

   (a) machine A (b) machine B.

Answers

1. $\mathsf{P}(A) = \dfrac{2}{16}$, $\mathsf{P}(B) = \dfrac{4}{16}$, $\mathsf{P}(C) = \dfrac{11}{16}$, $\mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) = \dfrac{17}{16}$

   A, B and C are not mutually exclusive since events A and C have outcomes in common. This is the reason why $\mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) = \dfrac{17}{16}$; we are adding the probabilities corresponding to common outcomes more than once.

2. (a) $\mathsf{P}(\text{person falls in the modal group}) = \dfrac{15}{50}$

   (b) $\mathsf{P}(\text{person has been working for less than 4 years}) = \dfrac{27}{50}$

   (c) $\mathsf{P}(\text{person has been working for more than 8 years}) = \dfrac{8}{50}$

3. $\mathsf{P}(A) \times \mathsf{P}(B) = \dfrac{1}{8}$ and $(1 - \mathsf{P}(A)) \times (1 - \mathsf{P}(B)) = \dfrac{3}{8}$

   Treat these equations as $xy = \dfrac{1}{8}$ and $(1-x)(1-y) = \dfrac{3}{8}$ and solve to get:

   $\mathsf{P}(A) = \dfrac{1}{2}$ (or $\dfrac{1}{4}$) and $\mathsf{P}(B) = \dfrac{1}{4}$ (or $\dfrac{1}{2}$)

4. (a) $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$ so $0.7 = 0.4 + p$ implying $p = 0.3$

   (b) $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A) \times \mathsf{P}(B)$ so $0.7 = 0.4 + p - 0.4 \times p$ implying $p = 0.5$.

Answers

5. $$\begin{aligned} \mathsf{P}((A \cap B) \cup (C \cap D)) &= \mathsf{P}(A \cap B) + \mathsf{P}(C \cap D) - \mathsf{P}(A \cap B \cap C \cap D) \\ &= p^2 + p^2 - p^4 \\ &= 2p^2 - p^4 \end{aligned}$$

6. Let A={first item chosen is defective}, B ={second item chosen is defective}

(a) $\mathsf{P}(A) = \dfrac{20}{100} = \dfrac{1}{5}$

(b) $\mathsf{P}(A \cap B) = \mathsf{P}(A|B)\mathsf{P}(A) = \dfrac{19}{99} \times \dfrac{20}{100} = \dfrac{19}{495}$

(c) $\mathsf{P}(B) = \mathsf{P}(B|A)\mathsf{P}(A) + \mathsf{P}(B|A')\mathsf{P}(A') = \dfrac{19}{99} \times \dfrac{20}{100} + \dfrac{20}{99} \times \dfrac{80}{100} = \dfrac{198}{990} = \dfrac{1}{5}$

7. Let A ={first spring chosen is faulty}, B ={second spring chosen is faulty}

(a) $\mathsf{P}(A) = \dfrac{3}{8}$

(b) $\mathsf{P}(B) = \mathsf{P}(B|A)\mathsf{P}(A) + \mathsf{P}(B|A')\mathsf{P}(A') = \dfrac{2}{7} \times \dfrac{3}{8} + \dfrac{3}{7} \times \dfrac{5}{8} = \dfrac{21}{56} = \dfrac{3}{8}$

8. Let A = {at least one Head occurs}, B = {two Heads occur}

$$\mathsf{P}(B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(A \cup B)} = \frac{\mathsf{P}(A) \times \mathsf{P}(B)}{\mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A) \times \mathsf{P}(B)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}} = \frac{1}{3}$$

9. Let A = {item from machine A}, B = {item from machine B}, D = {item is defective}.
We know that: $\mathsf{P}(A) = 0.1,\ \mathsf{P}(B) = 0.9,\ \mathsf{P}(D|A) = 0.01,\ \mathsf{P}(D|B) = 0.05.$

(a)

$$\begin{aligned} \mathsf{P}(A|D) &= \frac{\mathsf{P}(D|A)\mathsf{P}(A)}{\mathsf{P}(D|A)\mathsf{P}(A) + \mathsf{P}(D|B)\mathsf{P}(B)} \\ &= \frac{0.01 \times 0.1}{0.01 \times 0.1 + 0.05 \times 0.9} \\ &= 0.02 \end{aligned}$$

(b) Similarly $\mathsf{P}(B|D) = 0.98$